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M. J. Queijo
Langley Research Center, Hampton, Virginia
William R. Wells
Wright State University, Dayton, Ohio
Dinesh A. Keskar
University of Cincinnati, Cincinnati, Ohio

National Aeronautics
and Space Administration

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SUMMARY

An approximate indicial lift function associated with circulation has been developed for tapered, swept wings in incompressible flow. The function is derived by representing the wings with a simple vortex system. The results from the derived equations compare well with the limited available results from more rigorous and complex methods.

The equations, as derived, are not very convenient for calculating the dynamic response of aircraft, parameter extraction, or for determining frequency-response curves for wings. Therefore, an expression is developed to convert the indicial response function to an exponential form which is more convenient for these purposes. The exponential form is nearly as accurate as the form derived from the simplified vortex system.

INTRODUCTION

The indicial lift function following a unit step increase in wing angle of attack is fundamental to the calculation of airplane transient motion. The original derivation of this function (generally designated by $k_1(\bar{s})$) was developed in 1925 for two-dimensional incompressible flow by H. Wagner (ref. 1). Since that time the indicial response has been calculated for a number of specific wing planforms (refs. 2 to 4, for example) and vortex lattice methods are available for calculating indicial lift of arbitrary wing planform (ref. 5, for example). In addition, techniques are currently available for estimating the lift on wings undergoing oscillatory motions. (See refs. 6 to 9, for example.) Reference 10 contains a review and bibliography on work in unsteady aerodynamics.

Although available methods appear to be adequate for determining transient lift loads, they are generally complex and do not appear to be suitable for inclusion in aircraft motion studies (except for a few specific wing planforms).

The purpose of this paper is to develop a simplified method for determining the indicial lift function associated with circulation. The part associated with the "apparent additional mass" is not treated (eq. (31) of ref. 2). A second purpose is to provide an elementary approach to unsteady aerodynamics which, although it lacks the mathematical elegance of more exact theories, it does give the novice a starting point for study in an important and interesting area of aerodynamics.

SYMBOLS

A aspect ratio, b^2/S

a_0, a_1, a_2, b_1, b_2 constants

b	wing span
$\Delta C_L(t)$	three-dimensional indicial lift coefficient, a function of time
$\left(C_{L_\alpha}\right)_{ss}$	steady-state lift curve slope
c	wing chord
$c_l(0)$	section lift coefficient at $t = 0$
$\Delta c_l(t)$	two-dimensional indicial lift coefficient, a function of time
c_r	wing root chord
c_t	wing tip chord
i	$= \sqrt{-1}$
K	nondimensional constant
$k_1(\bar{s})$	Wagner function
L	wing lift
l	section lift, per unit span
P,Q,R	terms defined in equation (12)
S	wing area
s	Laplace variable
t	time
U	free-stream velocity
w	downwash velocity due to vortices
x_o	distance from three-quarter chord point of root chord to starting point of shed vortex along root-chord line
y,z	constants in exponential form of indicial response function
α	angle of attack
$\dot{\alpha}$	$= \dfrac{d\alpha}{dt}$
Γ	total strength of vortex
Γ_i	vortex strength of ith vortex ($i = 1, 2, \ldots, n$)

Λ sweep of wing quarter-chord line

Λ_s sweep of shed-vortex line

λ wing taper ratio, c_t/c_r

ρ air density

τ dummy variable

ω angular frequency

ANALYSIS

The basic concepts used in this study involve the use of lifting-line theory to model the change in lift following a unit step increase in angle of attack. The principles are derived for a two-dimensional wing and then extended to swept, tapered wings.

Two-Dimensional Wing

The two-dimensional wing is represented by a bound vortex located at the quarter-chord line. (See fig. 1(a).) The boundary condition of no flow through the wing is satisfied only at the wing three-quarter chord line. If the wing angle of attack α is suddenly given a unit increase, the bound (lifting) vortex strength increases by a value Γ. Since the total strength (or circulation) in the flow field must remain at the value it had before the unit increase in α, a counter vortex of strength $(-\Gamma)$ is formed near the wing trailing edge. The vortex strength is such that the normal velocity component at the wing three-quarter chord line, induced by the two vortices, is equal and opposite to the component of the free-stream velocity normal to the wing surface at the same point. As time increases, the shed vortex moves downstream, but retains its original strength. This movement would cause a decrease in downwash at the three-quarter chord point, but the decrease is offset by the shedding of additional vortices. (See fig. 1(b).) This means, in turn, that the bound lifting-line vortex strength increases in order to keep the total circulation unchanged. Consequently, there is a distribution of vorticity behind the wing. The vortex sheet is extending downstream at a rate equal to the free-stream velocity.

Wagner (ref. 1) obtained a solution for the lift following a unit step increase in α by applying a conformal transformation and satisfying the physical principle that the velocity at the wing trailing edge is finite at all time (Kutta condition). He derived for the lift, as a function of nondimensional distance $\bar{s} = Ut/(c/2)$, the expression

$$l = 2\pi\rho Uwk_1(\bar{s}) \tag{1}$$

Wagner did not derive an explicit analytical expression for $k_1(\bar{s})$ but gave only numerical values. Some years later (1936), Küssner (ref. 11) derived a long, slowly convergent series for $k_1(\bar{s})$.

The mathematical complexities of obtaining values or expressions for $k_1(\bar{s})$ arise because of the physical model depicted in figure 1(b), or its equivalent in the transformed-plane method of reference 1. The present report develops a simpler model and a simple, yet reasonably accurate, analytical expression for $k_1(\bar{s})$. The shed vortex sheet is replaced by a single shed vortex having the same (time varying) strength as the bound vortex and moving downstream at a velocity KU. The downwash velocity induced at the three-quarter chord point by the two vortices (bound and shed) is easily found by use of the Biot-Savart law (or see ref. 12, p. 127), and is

$$w = \frac{\Gamma}{2\pi}\left(\frac{1}{c/2} + \frac{1}{x_o + KUt}\right) \tag{2}$$

Solving for Γ results in

$$\Gamma = 2\pi w \frac{c}{2}\left(1 - \frac{1}{1 + \frac{x_o}{c/2} + \frac{KUt}{c/2}}\right) \tag{3}$$

The boundary condition is met by

$$w = U \sin \alpha$$

or, for small angles

$$w = U\alpha \tag{4}$$

Equations (3) and (4) can be used with the Kutta-Joukowski equation for section lift

$$l = \rho U \Gamma \tag{5}$$

to obtain an expression for the section lift coefficient due to a unit step increase in angle of attack

$$\Delta c_l(t) = 2\pi\left(1 - \frac{1}{1 + \frac{x_o}{c/2} + \frac{KUt}{c/2}}\right) \tag{6}$$

Under steady-state conditions ($t \to \infty$), $\Delta c_l = 2\pi$, which is the correct value. The distance x_o is related to the starting lift; that is, x_o is related to $c_l(0)$. From equation (6),

$$\Delta c_l(0) = 2\pi\left(1 - \frac{1}{1 + \frac{x_o}{c/2}}\right) \tag{7}$$

Using Wagner's conclusion that the starting lift is one-half of the steady-state lift leads, for the two-dimensional wing, to

$$\frac{x_o}{c/2} = 1 \tag{8}$$

Equation (6), therefore, becomes

$$\Delta c_l(t) = 2\pi\left(1 - \frac{1}{2 + \frac{KUt}{c/2}}\right) \tag{9}$$

The term in parentheses is the counterpart of the Wagner function $k_1(\bar{s})$. This term is found to approximate the Wagner function very closely if K equals 1/2. Equation (9) then becomes

$$\Delta c_l(t) = 2\pi\left(1 - \frac{1}{2 + \frac{1}{2}\frac{Ut}{c/2}}\right) \tag{10}$$

Garrick (ref. 13) arrived at exactly this same expression by curve-fitting the Wagner function.

The indicial response is approximated closely by use of the single bound-vortex line of figure 1(c) and the shed vortex (which represents the shed distribution of vorticity) starting at a distance $c/2$ behind the wing three-quarter chord line and moving downstream at one-half the free-stream velocity.

Three-Dimensional Wing

The simple vortex model of the two-dimensional wing compares well with Wagner's exact analysis; therefore, the same ideas have been tried on three-dimensional tapered, swept wings. There are several choices to be made relative to the mathematical model, and these choices include:

(1) At what point should the boundary condition be satisfied?

(2) Where should the shed vortex be started relative to the wing?

(3) How should the shed vortex be oriented relative to the wing?

In order to keep the model as simple as possible, it was decided to satisfy the boundary condition only at the three-quarter chord point of the root chord. The other two choices are made in a later section of this report.

The vortex model for the three-dimensional wing is indicated in figure 2. The downwash at the control point can be derived by using the Biot-Savart law (or see p. 127 of ref. 12) and is of the form

$$w = \frac{\Gamma}{\pi}\left(P + Q + R\right) \tag{11}$$

where

$$\left.\begin{aligned}
P &= \frac{1}{c_r \cos \Lambda}\left[\frac{\dfrac{b/2}{\cos \Lambda} - \dfrac{c_r}{2} \sin \Lambda}{\sqrt{\left(\dfrac{b/2}{\cos \Lambda} - \dfrac{c_r}{2} \sin \Lambda\right)^2 + \left(\dfrac{c_r}{2} \cos \Lambda\right)^2}} + \frac{\dfrac{c_r}{2} \sin \Lambda}{\sqrt{\left(\dfrac{c_r}{2} \sin \Lambda\right)^2 + \left(\dfrac{c_r}{2} \cos \Lambda\right)^2}}\right] \\
Q &= \frac{1}{b}\left[\frac{x_o + \dfrac{1}{2} Ut + \dfrac{b}{2} \tan \Lambda_s}{\sqrt{\left(x_o + \dfrac{1}{2} Ut + \dfrac{b}{2} \tan \Lambda_s\right)^2 + \left(\dfrac{b}{2}\right)^2}} + \frac{\dfrac{c_r}{2} - \dfrac{b}{2} \tan \Lambda}{\sqrt{\left(\dfrac{c_r}{2} - \dfrac{b}{2} \tan \Lambda\right)^2 + \left(\dfrac{b}{2}\right)^2}}\right] \\
R &= \frac{1}{2\left(x_o + \dfrac{1}{2} Ut\right) \cos \Lambda_s}\left\{\frac{\dfrac{b/2}{\cos \Lambda_s} + \left(x_o + \dfrac{1}{2} Ut\right) \sin \Lambda_s}{\sqrt{\left[\dfrac{b/2}{\cos \Lambda_s} + \left(x_o + \dfrac{1}{2} Ut\right) \sin \Lambda_s\right]^2 + \left[\left(x_o + \dfrac{1}{2} Ut\right) \cos \Lambda_s\right]^2}}\right. \\
&\qquad \left. - \frac{\left(x_o + \dfrac{1}{2} Ut\right) \sin \Lambda_s}{\sqrt{\left[\left(x_o + \dfrac{1}{2} Ut\right) \sin \Lambda_s\right]^2 + \left[\left(x_o + \dfrac{1}{2} Ut\right) \cos \Lambda_s\right]^2}}\right\}
\end{aligned}\right\} \tag{12}$$

The P,Q,R expressions are associated with the bound vortex, the wing-tip trailing-vortex pair, and the shed vortex, respectively. The geometric relationship $b/c_r = A(1 + \lambda)/2$ can be used to put equations (12) into more convenient forms

$$\left.\begin{aligned}
bP &= \frac{A(1+\lambda)}{2}\left\{\frac{\dfrac{A(1+\lambda)}{2}\sec^2\Lambda - \tan\Lambda}{\sqrt{\left[\dfrac{A(1+\lambda)}{2}\sec\Lambda - \sin\Lambda\right]^2 + \cos^2\Lambda}} + \tan\Lambda\right\} \\[2ex]
bQ &= \frac{\dfrac{x_o}{c_r/2} + \dfrac{1}{2}\dfrac{Ut}{c_r/2} + \dfrac{A(1+\lambda)}{2}\tan\Lambda_s}{\sqrt{\left[\dfrac{x_o}{c_r/2} + \dfrac{1}{2}\dfrac{Ut}{c_r/2} + \dfrac{A(1+\lambda)}{2}\tan\Lambda_s\right]^2 + \left[\dfrac{A(1+\lambda)}{2}\right]^2}} + \frac{1 - \dfrac{A(1+\lambda)\tan\Lambda}{2}}{\sqrt{\left[1 - \dfrac{A(1+\lambda)}{2}\tan\Lambda\right]^2 + \left[\dfrac{A(1+\lambda)}{2}\right]^2}} \\[2ex]
bR &= \frac{\dfrac{A(1+\lambda)}{2}}{\dfrac{x_o}{c_r/2} + \dfrac{1}{2}\dfrac{Ut}{c_r/2}}\left\{\frac{\dfrac{A(1+\lambda)}{2}\sec^2\Lambda_s + \left(\dfrac{x_o}{c_r/2} + \dfrac{1}{2}\dfrac{Ut}{c_r/2}\right)\tan\Lambda_s}{\sqrt{\left[\dfrac{A(1+\lambda)}{2}\sec\Lambda_s + \left(\dfrac{x_o}{c_r/2} + \dfrac{1}{2}\dfrac{Ut}{c_r/2}\right)\sin\Lambda_s\right]^2 + \left[\left(\dfrac{x_o}{c_r/2} + \dfrac{1}{2}\dfrac{Ut}{c_r/2}\right)\cos\Lambda_s\right]^2}} - \tan\Lambda_s\right\}
\end{aligned}\right\} \tag{13}$$

The lift of the wing is determined from the Kutta-Joukowski equation

$$L = \rho U \Gamma b \tag{14}$$

Nondimensionalizing and using equation (11) results in

$$C_L(t) = \frac{2\pi wb}{US(P + Q + R)} \tag{15}$$

The boundary condition requires that $w/U = \alpha$; hence, equation (15) becomes

$$C_L(t) = \frac{2\pi A\alpha}{b(P + Q + R)} \tag{16}$$

Under steady-state conditions ($t \to \infty$), equation (16), with proper substitution of equations (13), results in

$$\left(C_{L_\alpha}\right)_{ss} = \frac{2\pi A}{\left\{bP + \dfrac{1 - \dfrac{A(1 + \lambda)}{2} \tan \Lambda}{\sqrt{\left[1 - \dfrac{A(1 + \lambda)}{2} \tan \Lambda\right]^2 + \left[\dfrac{A(1 + \lambda)}{2}\right]^2}} + 1\right\}} \tag{17}$$

Solving for $2\pi A$ and substituting the result into equation (16) results in, for a unit step in α,

$$\Delta C_L(t) = \frac{\left(C_{L_\alpha}\right)_{ss}}{bP + bQ + bR}\left\{bP + \frac{1 - \dfrac{A(1 + \lambda)}{2} \tan \Lambda}{\sqrt{\left[1 - \dfrac{A(1 + \lambda)}{2} \tan \Lambda\right]^2 + \left[\dfrac{A(1 + \lambda)}{2}\right]^2}} + 1\right\} \tag{18}$$

As t becomes very large, equation (18) reduces to

$$\Delta C_L(t \to \infty) = \left(C_{L_\alpha}\right)_{ss}$$

However, equation (17) is not accurate because of the simple vortex-system representation of the wing. A concept proposed is to use the best available source for $\left(C_{L_\alpha}\right)_{ss}$ rather than equation (17). In this manner, attaining correct values for equation (18) is insured, at least for large values of time.

Before equation (18) can be used (with eqs. (13)), the terms $x_o/(c_r/2)$ must be determined, and a choice made for the sweep angle of the shed vortex. As noted previously, for a two-dimensional wing, a value of $x_o/(c_r/2) = 1.0$ is appropriate. This value is retained for simplicity so that the inboard ends of the shed vortex start at a distance $x_o/(c_r/2) = 1$ behind the three-quarter chord point of the root chord.

Selection of the sweep of the shed vortex was made by trying reasonable values; that is, (1) parallel to the wing trailing edge, and (2) parallel to the wing quarter-chord line. Comparison with results from more exact methods showed that the shed vortex sweep angle equal to that of the wing quarter-chord line was the better choice.

Based on these choices, equations (13) become

$$
\left.
\begin{aligned}
bP &= \frac{A(1+\lambda)}{2}\left\{\frac{\dfrac{A(1+\lambda)}{2}\sec^2\Lambda - \tan\Lambda}{\sqrt{\left[\dfrac{A(1+\lambda)}{2}\sec\Lambda - \sin\Lambda\right]^2 + \cos^2\Lambda}} + \tan\Lambda\right\} \\[2ex]
bQ &= \frac{1 + \dfrac{A(1+\lambda)}{2}\tan\Lambda + \dfrac{1}{2}\dfrac{Ut}{c_r/2}}{\sqrt{\left[1 + \dfrac{1}{2}\dfrac{Ut}{c_r/2} + \dfrac{A(1+\lambda)}{2}\tan\Lambda\right]^2 + \left[\dfrac{A(1+\lambda)}{2}\right]^2}} + \frac{1 - \dfrac{A(1+\lambda)}{2}\tan\Lambda}{\sqrt{\left[1 - \dfrac{A(1+\lambda)}{2}\tan\Lambda\right]^2 + \left[\dfrac{A(1+\lambda)}{2}\right]^2}} \\[2ex]
bR &= \frac{\dfrac{A(1+\lambda)}{2}}{1 + \dfrac{1}{2}\dfrac{Ut}{c_r/2}}\left\{\frac{\dfrac{A(1+\lambda)}{2}\sec^2\Lambda + \left(1 + \dfrac{1}{2}\dfrac{Ut}{c_r/2}\right)\tan\Lambda}{\sqrt{\left[\dfrac{A(1+\lambda)}{2}\sec\Lambda + \left(1 + \dfrac{1}{2}\dfrac{Ut}{c_r/2}\right)\sin\Lambda\right]^2 + \left[\left(1 + \dfrac{1}{2}\dfrac{Ut}{c_r/2}\right)\cos\Lambda\right]^2}} - \tan\Lambda\right\}
\end{aligned}
\right\}
\tag{19}
$$

Equations (18) and (19), with suitable selection of an expression for $\left(C_{L_\alpha}\right)_{ss}$ (use of ref. 14 or 15, for example), are the basic equations for the indicial response for tapered, swept wings in incompressible flow. Although the equations are somewhat lengthy, several factors appear repeatedly and are constant during the lift buildup.

LIFT VARIATION FOR ARBITRARY CHANGE IN ANGLE OF ATTACK

The lift of a wing undergoing an arbitrary change in angle of attack can be determined by use of the indicial response function in Duhamel's integral; so that

$$C_L(t) = \int_0^t \left[\Delta C_L(t - \tau)\ \dot{\alpha}(\tau)\right] d\tau \tag{20}$$

Frequency response characteristics can be obtained easily by using the Laplace transform of equation (20) so that

$$C_L(s) = \Delta C_L(s)\ \dot{\alpha}(s) = \left[\Delta C_L(s)\right](s)\left[\alpha(s)\right] \tag{21}$$

from which

$$C_L(i\omega) = \left[\Delta C_L(i\omega)\right](i\omega)\left[\alpha(i\omega)\right]$$

Because of the manner in which time enters into equations (18) and (19), closed-form Laplace transforms are not known to exist for the general indicial lift function $\Delta C_L(t)$. Therefore, a form for which closed-form Laplace transforms do exist, which would also be convenient, has been sought. In reference 2 the indicial response for several specific elliptic wings were given in exponential series of the form

$$\Delta C_L = a_0 + a_1 e^{-b_1\left[Ut/(c_r/2)\right]} + a_2 e^{-b_2\left[Ut/(c_r/2)\right]} + \ldots \tag{22}$$

In the present study, equation (18) was approximated by just one exponential term, namely,

$$\frac{\Delta C_L(t)}{\left(C_{L\alpha}\right)_{ss}} = 1 - ye^{-z\left[Ut/(c_r/2)\right]} \tag{23}$$

that leads to

$$\frac{\Delta C_L(t)}{\left(C_{L\alpha}\right)_{ss}} = \frac{1}{bP + bQ + bR}\left\{bP + 1 + \frac{1 - \frac{A(1 + \lambda)}{2}\tan\Lambda}{\sqrt{\left[1 - \frac{A(1 + \lambda)}{2}\tan\Lambda\right]^2 + \left[\frac{A(1 + \lambda)}{2}\right]^2}}\right\}$$

$$= 1 - ye^{-z\left[Ut/(c_r/2)\right]} \tag{24}$$

The parameter y was determined by evaluating equation (24) at $Ut/(c_r/2) = 0$ and solving for y. The resulting equation is

$$y = 1 - \frac{1}{\left[bP + bQ + bR\right]_{Ut/(c_r/2)=0}}\left\{bP + 1 + \frac{1 - \frac{A(1 + \lambda)}{2}\tan\Lambda}{\sqrt{\left[1 - \frac{A(1 + \lambda)}{2}\tan\Lambda\right]^2 + \left[\frac{A(1 + \lambda)}{2}\right]^2}}\right\} \tag{25}$$

where

$$
\left.
\begin{aligned}
(bP) &= \frac{A(1+\lambda)}{2}\left\{\frac{\frac{A(1+\lambda)}{2}\sec^2\Lambda - \tan\Lambda}{\sqrt{\left[\frac{A(1+\lambda)}{2}\sec\Lambda - \sin\Lambda\right]^2 + \cos^2\Lambda}} + \tan\Lambda\right\} \\
(bQ)_{Ut/(c_r/2)=0} &= \frac{1 + \frac{A(1+\lambda)}{2}\tan\Lambda}{\sqrt{\left[1 + \frac{A(1+\lambda)}{2}\tan\Lambda\right]^2 + \left[\frac{A(1+\lambda)}{2}\right]^2}} \\
&\quad + \frac{1 - \frac{A(1+\lambda)}{2}\tan\Lambda}{\sqrt{\left[1 - \frac{A(1+\lambda)}{2}\tan\Lambda\right]^2 + \left[\frac{A(1+\lambda)}{2}\right]^2}} \\
(bR)_{Ut/(c_r/2)=0} &= \frac{A(1+\lambda)}{2}\left\{\frac{\frac{A(1+\lambda)}{2}\sec^2\Lambda + \tan\Lambda}{\sqrt{\left[\frac{A(1+\lambda)}{2}\sec\Lambda + \sin\Lambda\right]^2 + \cos^2\Lambda}} - \tan\Lambda\right\}
\end{aligned}
\right\}
\quad (26)
$$

Determination of the parameter z is more difficult. If equation (23) is solved for z, the result is

$$
z = -\frac{1}{Ut/(c_r/2)}\left\{\log_e \frac{1}{\gamma}\left[1 - \frac{\Delta C_L(t)}{\left(C_{L_\alpha}\right)_{ss}}\right]\right\} \quad (27)
$$

Evaluation of equation (27) at $Ut/(c_r/2) = 0$ yields the indeterminant form 0/0, since y is precisely equal to $\left[1 - \frac{\Delta C_L(0)}{(C_{L_\alpha})_{ss}}\right]$. However L'Hospital's rule can be used with equation (27) and the resulting expression evaluated for z. By using L'Hospital's rule

$$z = - \lim_{Ut/(c_r/2)\to 0} \left\{ \frac{1}{y} \frac{\dfrac{d}{d\left(\dfrac{Ut}{c_r/2}\right)}\left[1 - \dfrac{\Delta C_L(t)}{(C_{L_\alpha})_{ss}}\right]}{\dfrac{1}{y}\left[1 - \dfrac{\Delta C_L(0)}{(C_{L_\alpha})_{ss}}\right]} \right\}$$

$$= - \lim_{Ut/(c_r/2)\to 0} \left\{ \frac{1}{y} \frac{d}{d\left(\dfrac{Ut}{c_r/2}\right)}\left[1 - \frac{\Delta C_L(t)}{(C_{L_\alpha})_{ss}}\right] \right\}$$

Using equation (24) results in

$$z = \frac{1}{y}\left\{ bP + 1 + \frac{1 - \dfrac{A(1+\lambda)}{2}\tan\Lambda}{\sqrt{\left[1 - \dfrac{A(1+\lambda)}{2}\tan\Lambda\right]^2 + \left[\dfrac{A(1+\lambda)}{2}\right]^2}} \right\}$$

$$\times \left[\frac{d}{d\left(\dfrac{Ut}{c_r/2}\right)} \frac{1}{bP + bQ + bR}\right]_{Ut/(c_r/2)=0}$$

or

$$z = -\frac{1}{y}\left\{bP + 1 + \frac{1 - \dfrac{A(1+\lambda)}{2}\tan\Lambda}{\sqrt{\left[1 - \dfrac{A(1+\lambda)}{2}\tan\Lambda\right]^2 + \left[\dfrac{A(1+\lambda)}{2}\right]^2}}\right\}$$

$$\times\left[\frac{1}{(bP + bQ + bR)^2}\right]_{Ut/(c_r/2)=0}\left[\frac{d}{d\left(\dfrac{Ut}{c_r/2}\right)}(bP + bQ + bR)\right]_{Ut/(c_r/2)=0} \qquad (28)$$

The derivatives appearing in equation (28) were evaluated, and the results are

$$\left.\begin{aligned}
\left[\frac{d(bP)}{d\left(\dfrac{Ut}{c_r/2}\right)}\right]_{Ut/(c_r/2)=0} &= 0 \\
\left[\frac{d(bQ)}{d\left(\dfrac{Ut}{c_r/2}\right)}\right]_{Ut/(c_r/2)=0} &= \frac{\dfrac{1}{2}\left[\dfrac{A(1+\lambda)}{2}\right]^2}{\left\{\left[1 + \dfrac{A(1+\lambda)}{2}\tan\Lambda\right]^2 + \left[\dfrac{A(1+\lambda)}{2}\right]^2\right\}^{3/2}} \\
\left[\frac{d(bR)}{d\left(\dfrac{Ut}{c_r/2}\right)}\right]_{Ut/(c_r/2)=0} &= -\frac{1}{2}(bR)_{Ut/(c_r/2)=0} + \frac{A}{4}\,\frac{(1+\lambda)\tan\Lambda}{\sqrt{\left[\dfrac{A(1+\lambda)}{2}\sec\Lambda + \sin\Lambda\right]^2 + \cos^2\Lambda}} - \frac{A(1+\lambda)}{4} \\
&\quad\times\frac{\left[\dfrac{A(1+\lambda)}{2}\sec^2\Lambda + \tan\Lambda\right]\left\{\left[\dfrac{A(1+\lambda)}{2}\sec\Lambda + \sin\Lambda\right]\sin\Lambda + \cos^2\Lambda\right\}}{\left\{\left[\dfrac{A(1+\lambda)}{2}\sec\Lambda + \sin\Lambda\right]^2 + \cos^2\Lambda\right\}^{3/2}}
\end{aligned}\right\} \qquad (29)$$

Equations (25) and (26) were used to calculate y for a wide range of wing sweep angles, aspect ratio, and taper ratio. Equations (28), (26), and (29) were used to calculate z for the same range of geometric variables. Results from the calculations are presented in figure 3 and can be used in equation (23) for calculating indicial response.

COMPARISON OF RESULTS WITH OTHER METHODS

This section compares results from the methods developed in this study with other methods for which numerical results are published.

Two-Dimensional Wing

The two-dimensional wing is a rather special case having $A \to \infty$, $\lambda = 1$, $\Lambda = 0$. The indicial-lift function given by the equation is relatively simple (eq. (10)) and does possess a closed-form Laplace transform. There is no need, therefore, to use the exponential form for the indicial lift. Results from equation (10) are compared with those of references 1, 2, 4, 5, and 11 in figure 4. The results from equation (10) are in excellent agreement with those of references 1, 2, 4, and 11, which are exact solutions or numerical approximations of exact solutions. The success of equation (10) in matching these exact solutions is attributable to the selection of x_o (the starting point of the shed vortex) and the velocity of the shed vortex to match the Wagner solution.

Three-Dimensional Wings

Vortex systems and doublet distributions used to model three-dimensional wings are generally rather complex and lead to involved calculations for determining indicial lift. Numerical results, therefore, are available for only a few specific wings for comparison with equation (23).

Rectangular wings.- Rectangular wings have $\lambda = 1.0$ and $\Lambda = 0$. Results from use of equation (23), with values of y and z from figure (3), are compared with results from other more exact (and more complex) methods in figure 5 for wings of aspect ratio 6 and 4. The figure shows that the approximate method of the present paper is in good agreement with the results of references 4 and 5.

Triangular wings.- Triangular wings have $\lambda = 0$ and, in addition, the wing geometry leads to a simple relationship between aspect ratio and the angle of the quarter-chord line, that is,

$$\tan \Lambda = \frac{3}{A} \tag{30}$$

Results from use of equation (23) and values of y and z from figure 3 are compared with results of reference 16 in figure 6 for triangular wings of aspect ratio 4 and 2. In these cases the agreement between the results of the present approximate theory with results of reference 16 is excellent.

Limiting case with $A \to 0$.- The parameter y for wings having aspect ratio of zero can be determined readily from equations (25) and (26) and is

$$y_{A \to 0} = 0 \tag{31}$$

The indicial response function, therefore, is

$$\left[\frac{\Delta C_L(t)}{\left(C_{L_\alpha}\right)_{ss}}\right]_{A \to 0} = 1.0 \tag{32}$$

This result agrees with the results of reference 16.

CONCLUDING REMARKS

An approximate indicial lift function associated with circulation has been developed for tapered, swept wings in incompressible flow. The function is derived based on representing the wings by a simple vortex system. Comparison of results from the derived equations compare well with the limited available results from more rigorous and complex methods.

The equations, as derived, are not very convenient for calculating the dynamic response of aircraft, parameter extraction, or for determining frequency-response curves for wings. An expression, therefore, is developed to convert the indicial response function to an exponential form which is more convenient for these purposes. The exponential form is nearly as accurate as the form derived from the simplified vortex system.

Langley Research Center
National Aeronautics and Space Administration
Hampton, VA 23665
May 5, 1978

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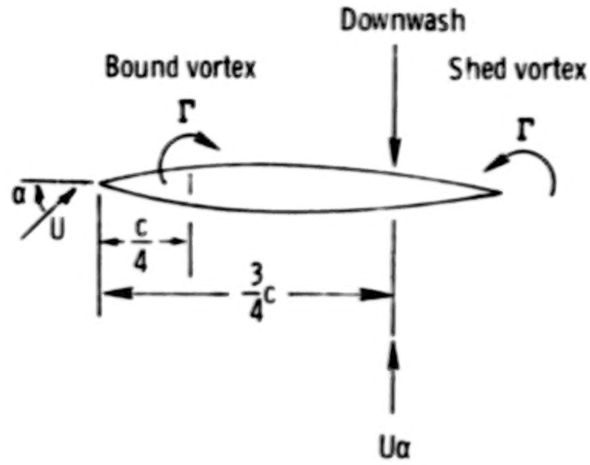
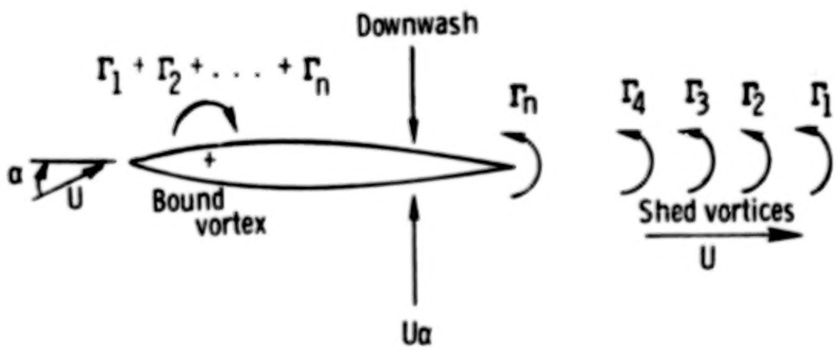
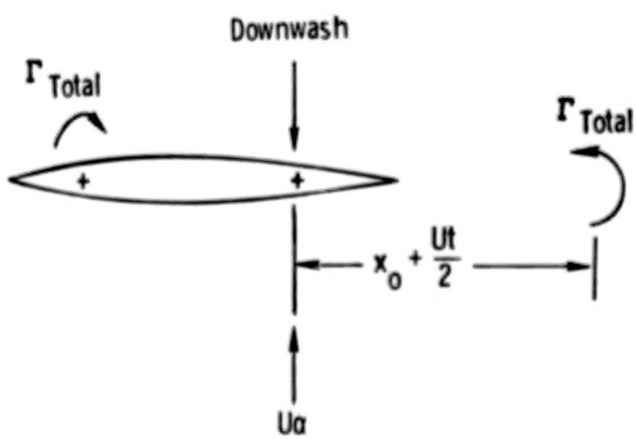


(a) Vortices at instant after α is increased by a step.



(b) Vortices some finite time after a step increase in α.



(c) Simplified vortex system used in this study.

Figure 1.- Vortex representation of a two-dimensional wing following a step increase in angle of attack.

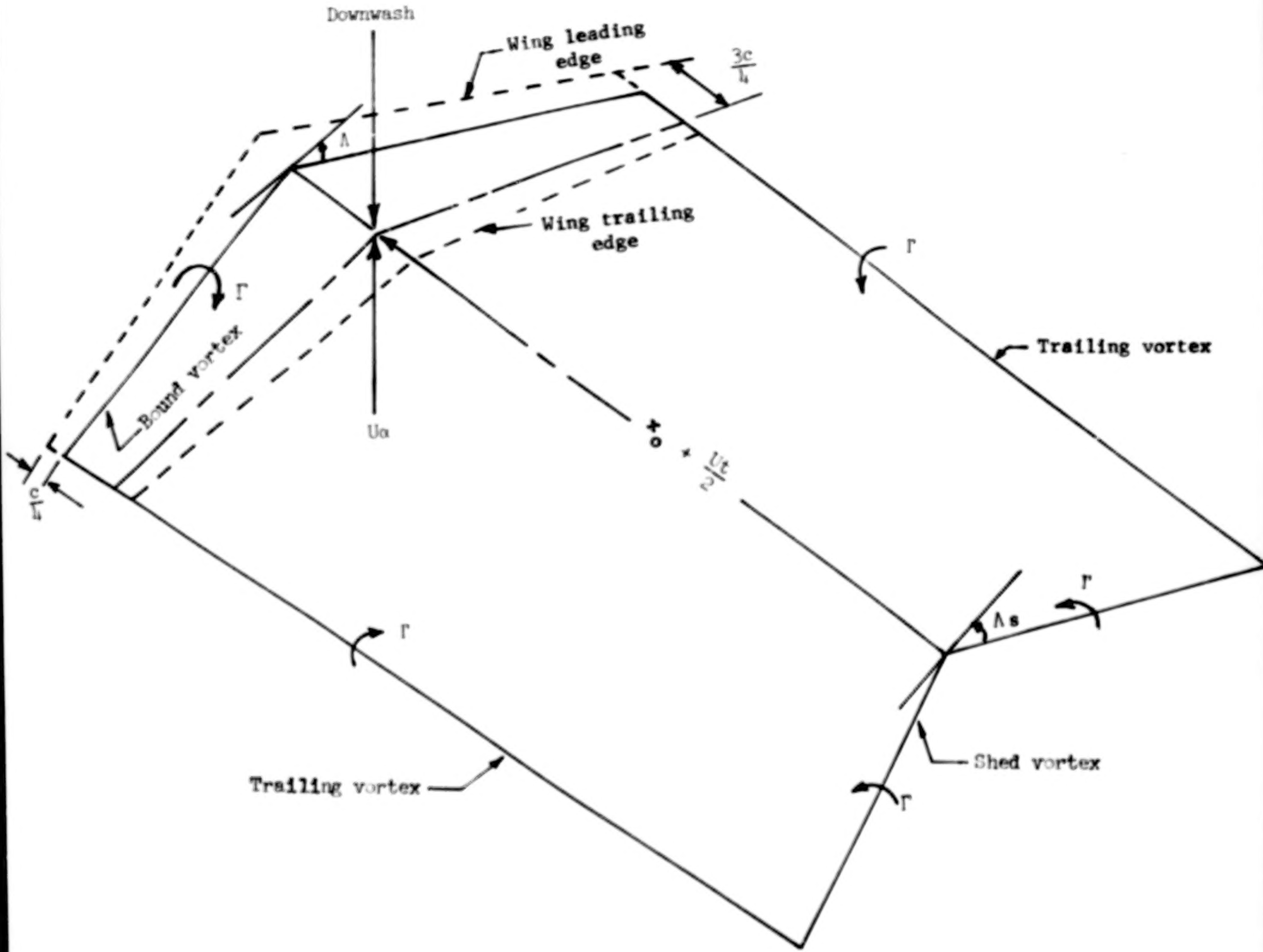
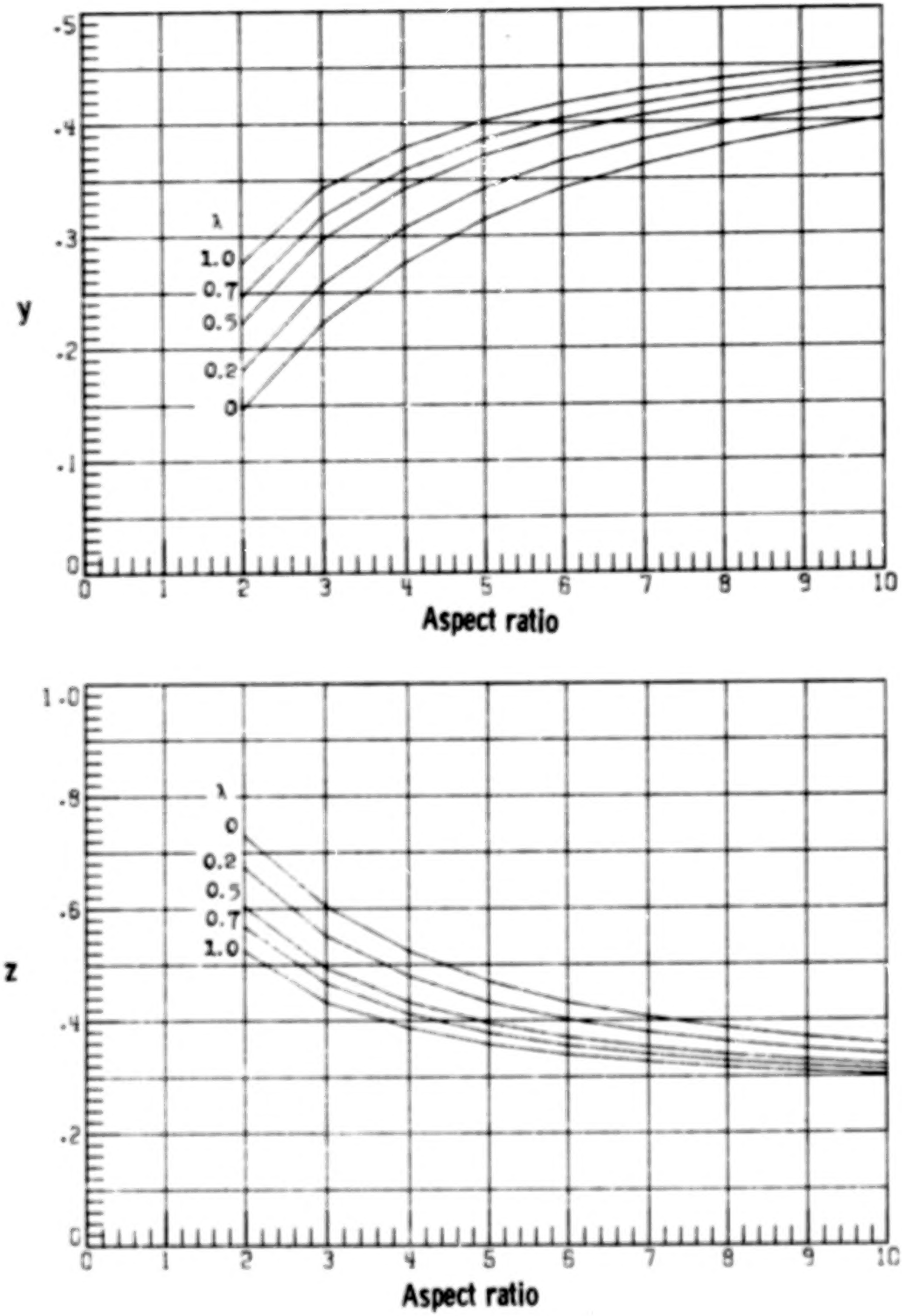


Figure 2.- Vortex system for swept tapered wings.



(a) $\Lambda = 0^{\circ}$.

Figure 3.- Values of constants y and z for use in the indicial response function $\Delta C_L / \left(C_{L_\alpha}\right)_{ss} = 1 - ye^{-z\left[Ut/(c_r/2)\right]}$

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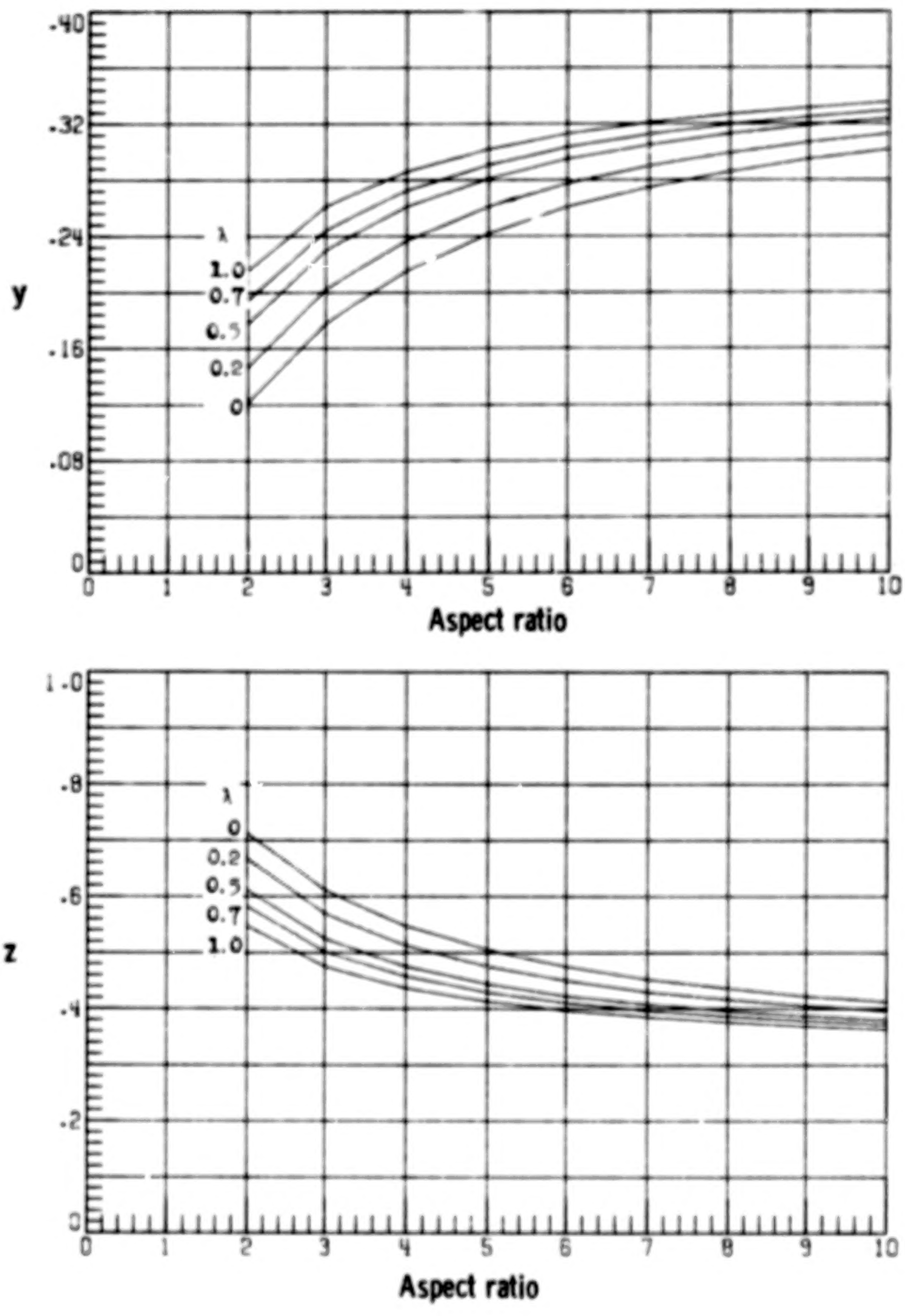
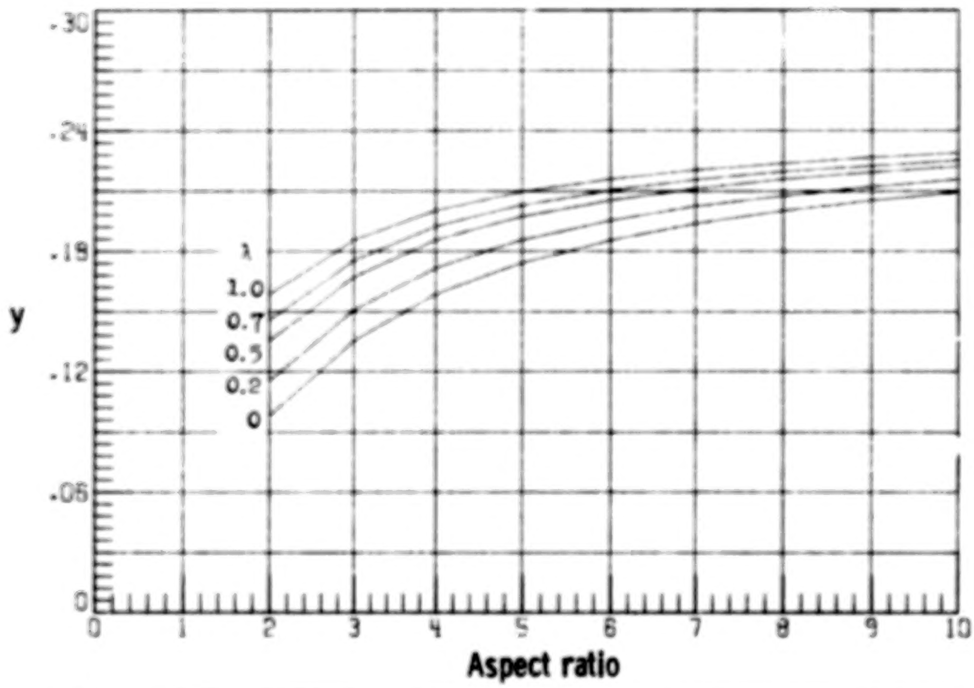
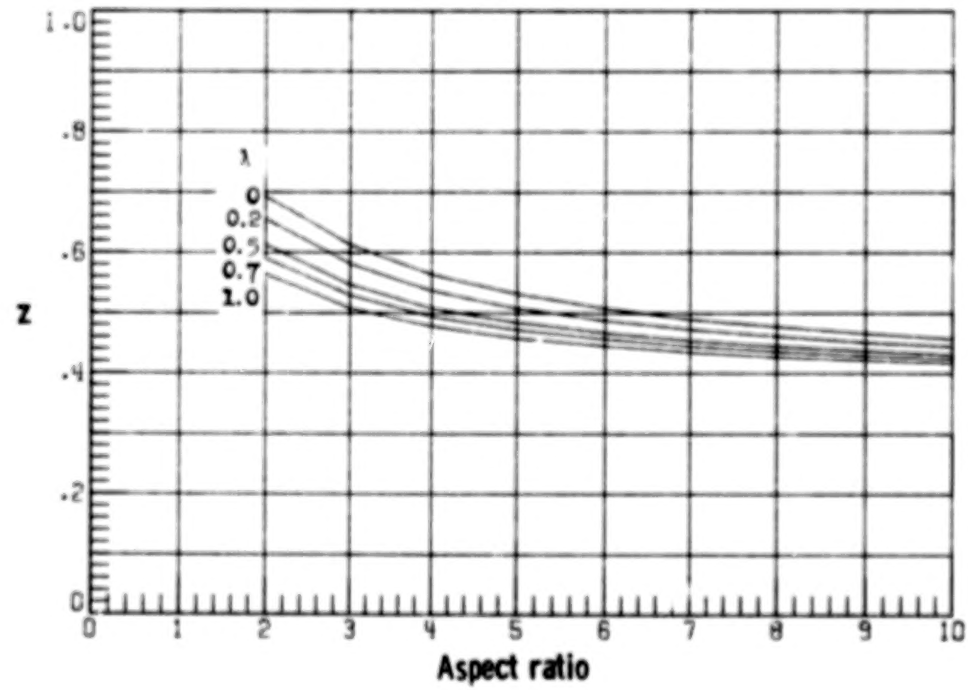


(b) $\Lambda = 15^{o}$.

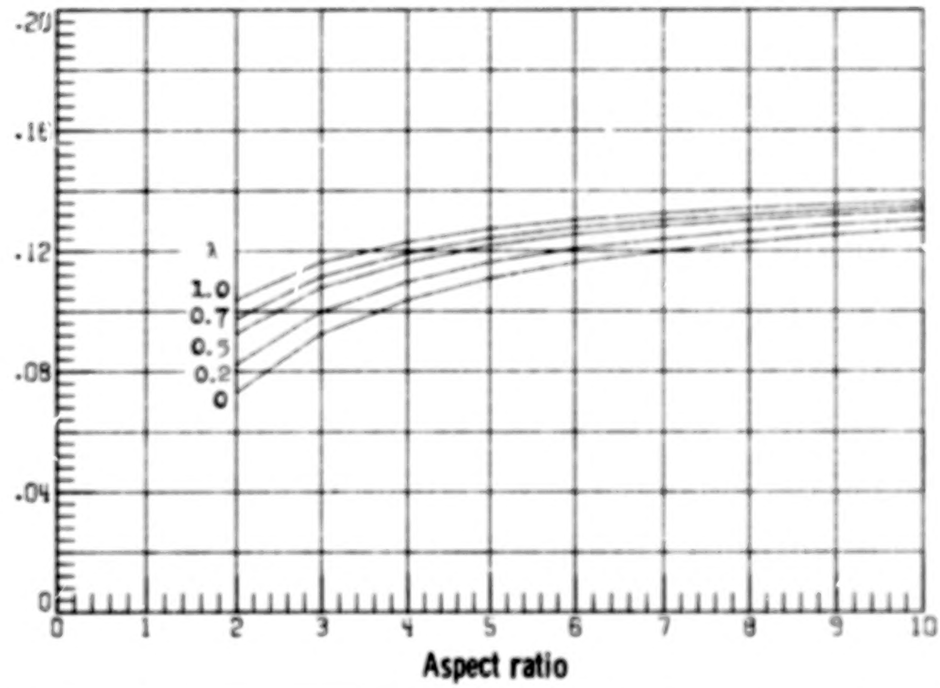
Figure 3.- Continued.

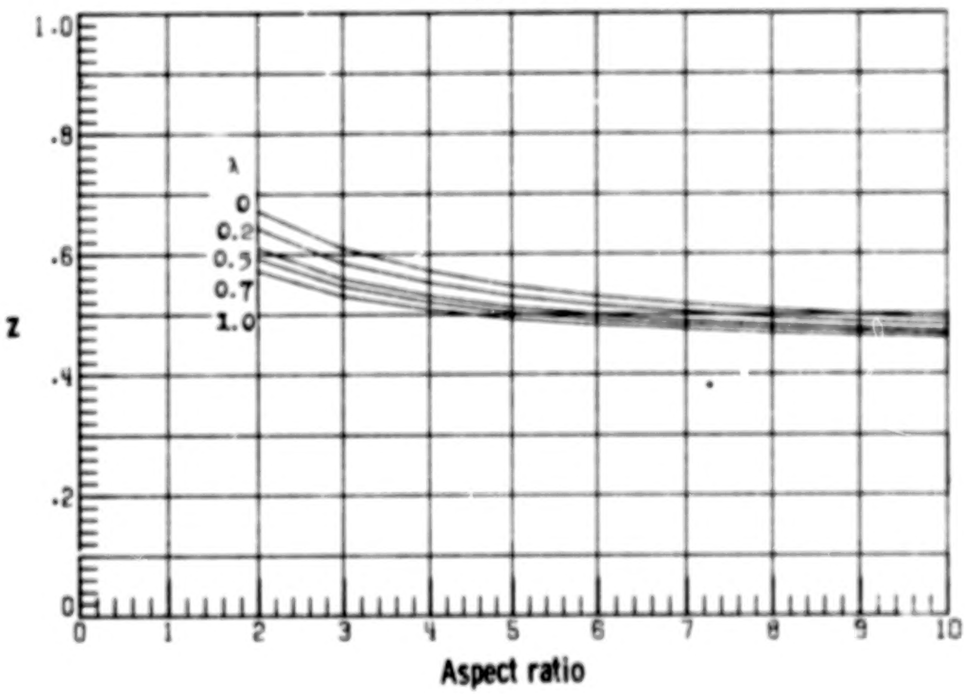




(c) $\Lambda = 30^{\circ}$.

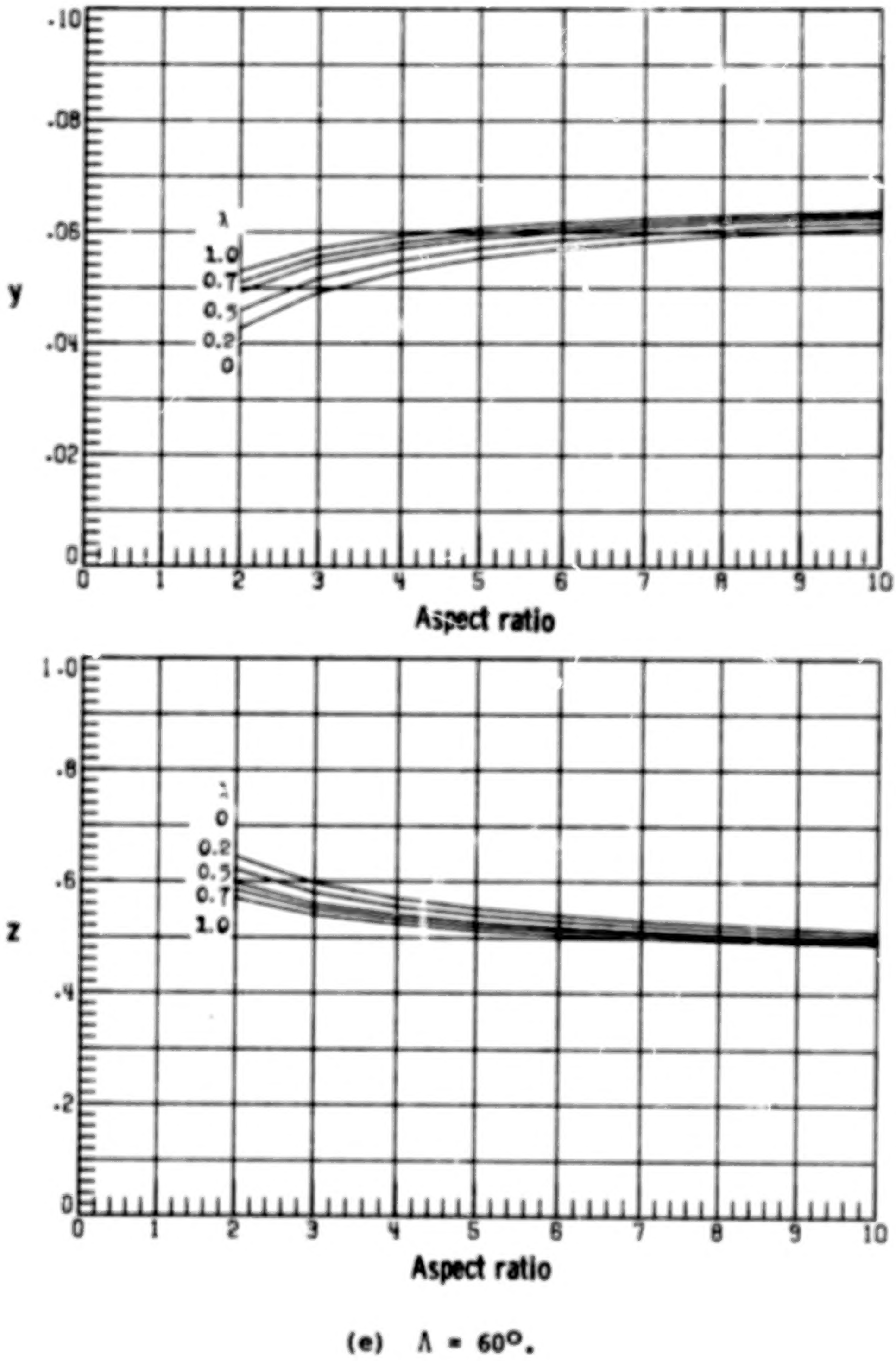
Figure 3.- Continued.





(d) $\Lambda = 45^{\circ}$.

Figure 3.- Continued.



(e) $\Lambda = 60^{\circ}$.

Figure 3.- Concluded.

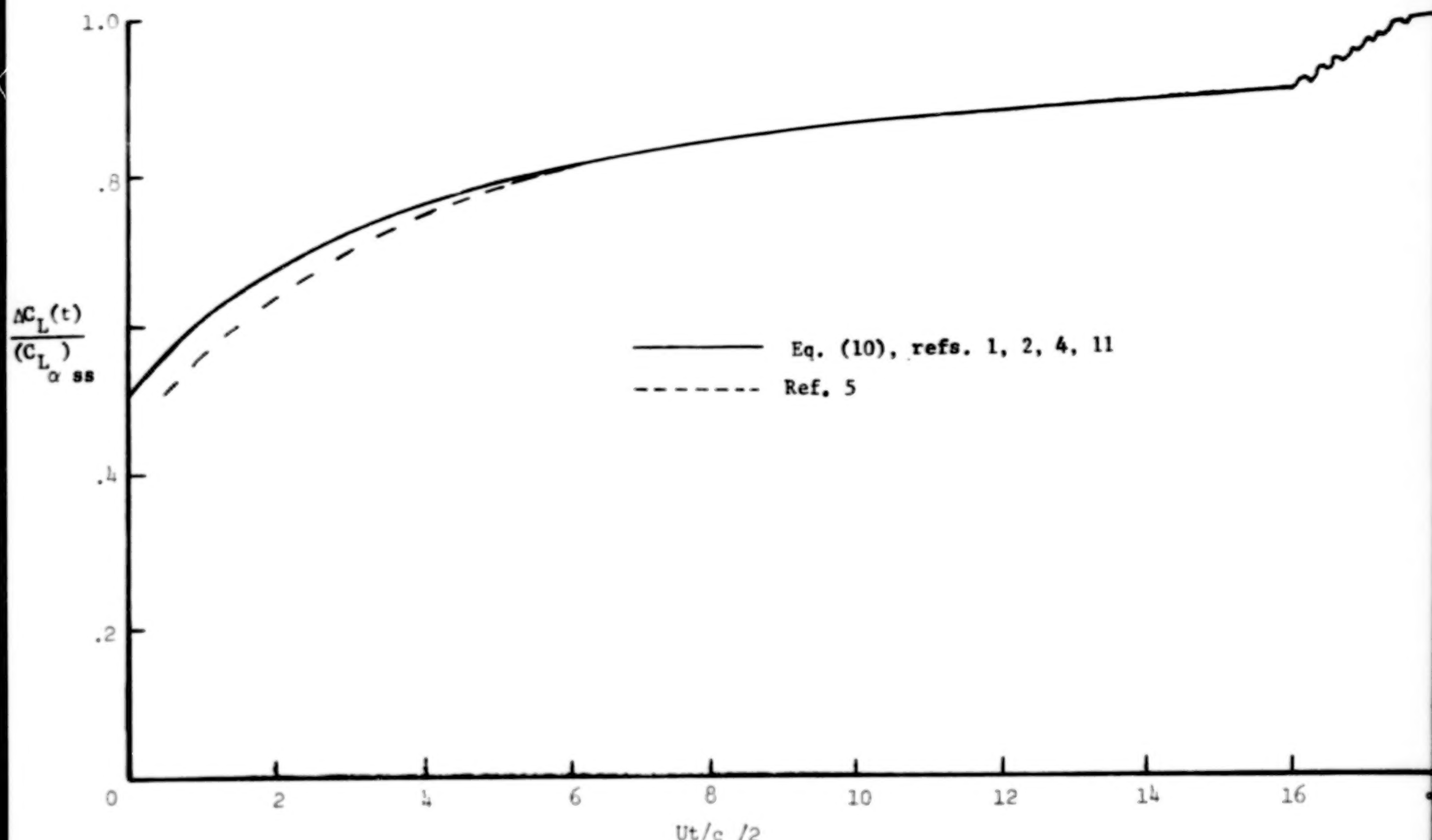
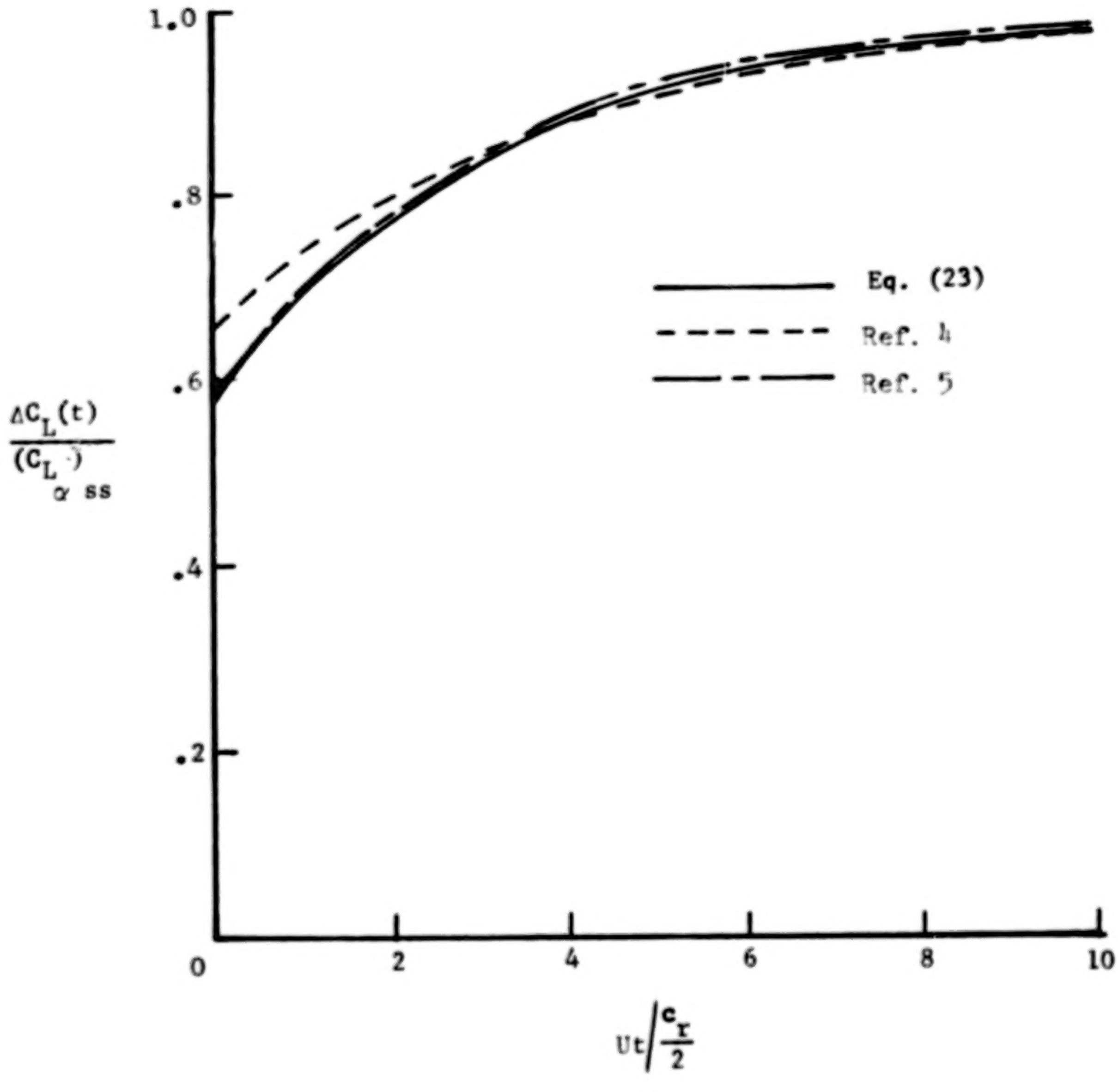
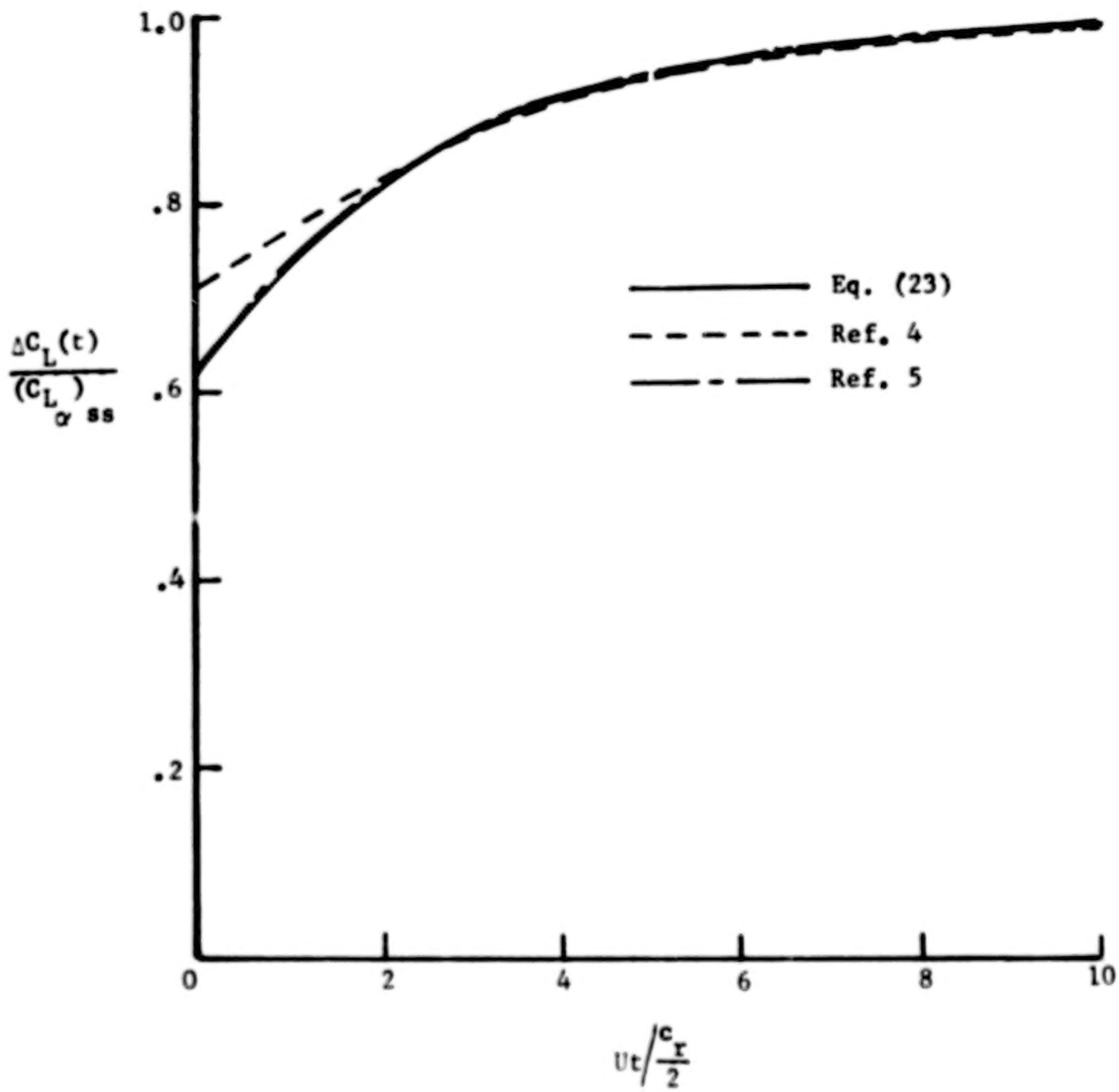


Figure 4.- Indicial lift for two-dimensional wings.



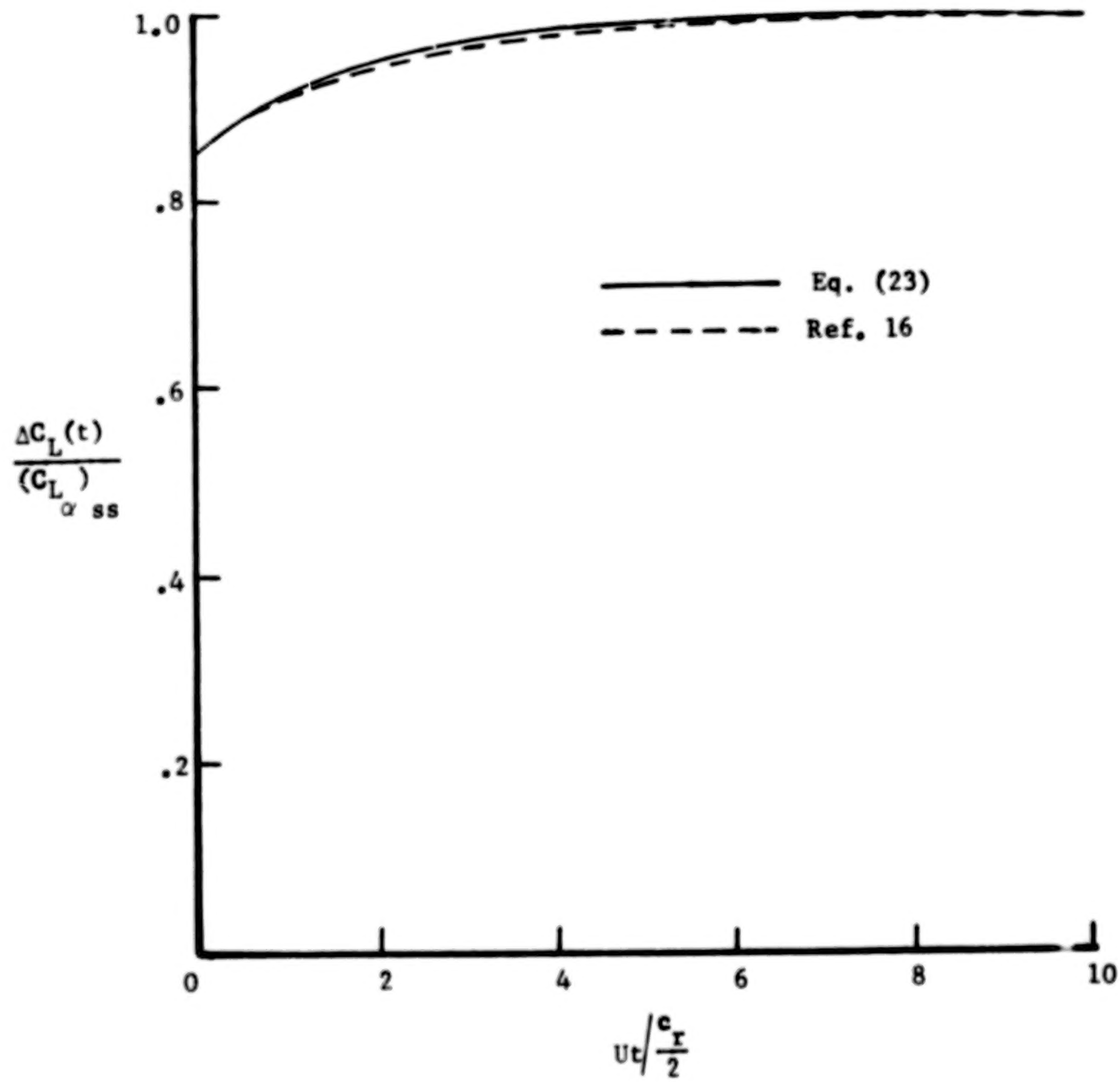
(a) Aspect ratio 6.0.

Figure 5.- Indicial lift for unswept, untapered wings.



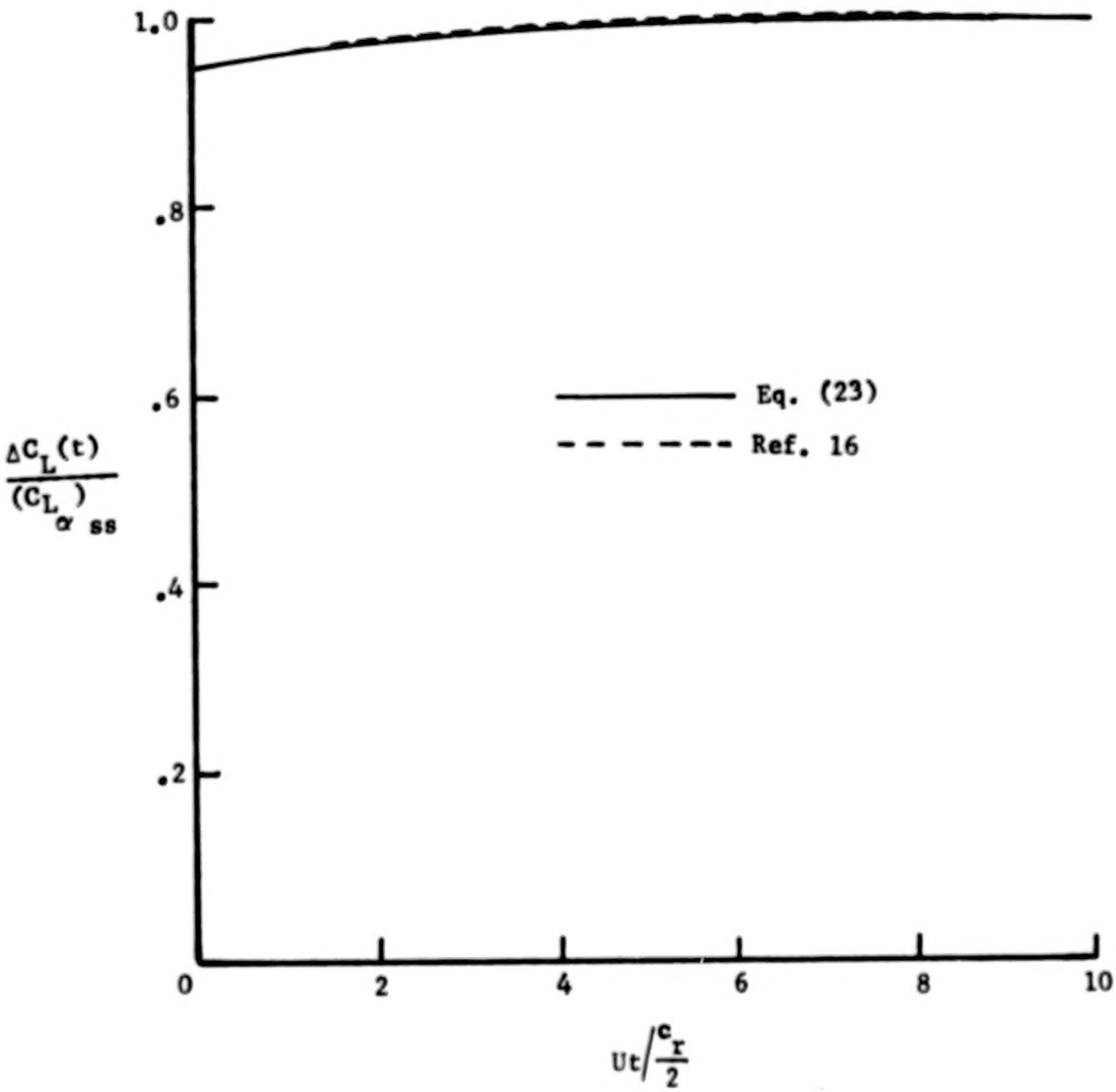
(b) Aspect ratio 4.0.

Figure 5.- Concluded.



(a) Aspect ratio 4.0.

Figure 6.- Indicial lift for delta (triangular) wings.



(b) Aspect ratio 2.0.

Figure 6.- Concluded.

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15. Supplementary Notes

M. J. Queijo: Langley Research Center, Hampton, Virginia.
William R. Wells: Wright State University, Dayton, Ohio.
Dinesh A. Keskar: University of Cincinnati, Cincinnati, Ohio.

16. Abstract

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